Journal of Finance - Marketing Research; Vol. 3, Issue 1; 2025
p-ISSN: 3030-4296; e-ISSN: 3030-430X
DOI: https://doi.org/10.52932/jfm.v3i1e

Journal of Finance - Marketing Research

http://jfm.ufm.edu.vn

A CLASS OF ECONOMIC GROWTH MODEL WITH MEMORY IN THE CONTEXT OF FRACTIONAL DERIVATIVES

Tran Dinh Phung[1], Nguyen Van Phong[1]*
[1]University of Finance - Marketing,Vietnam

ARTICLE INFO

DOI:
10.52932/jfm.v3i1e.670

Received:
November 12, 2024
Accepted:
February 17, 2025
Published:
March 25, 2025

Keywords:
Analytical solution, Economic growth model, Fractional derivatives, Fractional differential equation.

JEL codes:
C00, C02, E20

ABSTRACT

This article presents a model of economic growth that considers the effects of generalized power-law fading memory. To consider the memory effects in macroeconomic models, we propose a class of fractional differential equations involving a Caputo-type fractional derivative with respect to another function (or ψ-Caputo derivative). For this purpose, the objective of this paper is to obtain the unique solution of the proposed ψ-Caputo fractional differential equation. Behaviors of Mittag-Leffler's functions, which characterize the growth behaviors of the solution are described. To evaluate the solution, we will compute the Mittag-Leffler functions by Matlab code. As a result, we obtain the solutions in explicit form. This work also provides a comparative analysis of solutions of our model and model without memory effect. To the best of our knowledge, the Harrod-Domar fractional differential equation with Caputo-type fractional derivative has not yet been investigated. Hence the result obtained from our study is essentially new. Moreover, this approach allows for greater flexibility in modeling economic behavior, making it possible to account for non-instantaneous effects and the cumulative impact of previous decisions.

**Corresponding author:*
Email: nv.phongbmt@ufm.edu.vn

1. Introduction

It is well-known that fractional derivative is a classical mathematical concept with a long story as traditional derivative. A differential equation is fractional if it involves an operator that can be considered to be between a $(k-1)^{th}$ - and k^{th} - order differential operator, for some positive integer k, and it is said to be a fractional-order differential equation if this operator is the highest order operator in the equation.

The fractional differential equation is a convenient tool that describes processes with memory in physical sciences and economic theory. Many investigations in the economic model in the context of fractional calculus exist. The pioneering work for this type of problem is due to Granger and Joyeux (1980), who used fractional differencing and integrating to investigate the economic processes with memory. These fractional and integral calculus operations were proposed one hundred and fifty years ago and are used in economics without directly relating to the well-known fractional and finite difference calculus of non-integer order.

Recently, Tarasova and Tarasov (2017, 2018) have used fractional calculus to describe the power-law memory and then proposed the generalization of several macroeconomic models with continuous time such as the natural growth model, the Keynes model, the Harrod-Domar model, which play an important role in studying economic phenomena and processes.

Traore and Sene (2020) revisited the Ramsey model in the context of fractional calculus, the idea was to replace the first-order derivative with the fractional-order derivative and then investigate the possible impact of memory on economic growth.

Many linear or nonlinear economic models with fading memory have been studied, for example, the Haavelmo model with memory (Tarasov, 2021), the Evans model with memory (Tarasov, 2020), economic growth models with fractional derivative (Tejado et al., 2019). References to other works akin to economic models with memory can be found in the papers of Tarasova and Tarasov (2018), and Kee et al. (2022).

The aim of this paper is to study an economic growth model in the context of fractional calculus. Recall that the classical economic growth model of Harrod-Domar reads

$$\frac{dY(t)}{dt} = \frac{1}{v} I(t), \qquad t > 0 \tag{1.1}$$

where v is a positive constant, called the investment ratio and describes the rate of acceleration, the marginal productivity from capital (acceleration rate), $I(t)$ is the net investment function, $Y(t)$ is the value of output at time t, and $\frac{dY(t)}{dt} = Y'(t)$ is the first-order derivative against time t of $Y(t)$, that describes the growth rate of the output $Y(t)$.

We will consider the Harrod-Domar model, in which the ordinary derivative is replaced by a generalized form of the Caputo fractional derivative.

2. Literature review and hypothesis development

2.1. Literature review

It is known that differentiation and integration are the most important tools for constructing economic phenomena and modeling economic processes. Because the behavior of economic agents may be based on past fluctuations in the economy, then using fractional derivatives instead of classical derivative will take advantage of the memory effect features, allowing for historical observation of the economy. Many definitions of the fractional-order derivative have been proposed. For example, Grünwald-Letnikov, Caputo, Riemann-Liouville, Weyl, Hadamard, ψ-Caputo derivatives etc. It is worth mentioning here that fractional derivative operator has "non-local" property in that a fractional derivative has memory, which makes the fractional derivatives different from the classical derivative, where the standard derivative is a "local" operator, it is uniquely determined at the point t. In the field of economics, the application of fractional calculus in economic models has only recently attracted considerable research attention.

We recall the standard Harrod-Domar model with continuous time under the assumption of $I(t)$ as follows: the net investment value is a fixed part of the profit proportional to the difference between incomev $pY(t)$ and cost $C(t)$ (Tarasova & Tarasov, 2016):

$$I(t) = m(pY(t) - C(t)) \tag{2.1}$$

where m is the net investment rate ($0 < m < 1$), that is the profit-sharing used for the net investment. Here we further assume that the cost $C(t)$ is a linear function, that is $C(t) = aY(t) + b$ where a is the marginal cost, that is part of the cost that depends on the value of the output, while b is an independent cost that is part of the cost that does not depend on the value of the output.

Combing (1.2) and (1.1), we get

$$Y'(t) - \frac{m(p-a)}{v} Y(t) = -\frac{mb}{v} \tag{2.2}$$

The differential equation (1.3) describes the dynamics of the output $Y(t)$ within the framework of the Harrod-Domar model. This equation points out that the dynamics of $Y(t)$ are determined by the behavior of the function $C(t)$ (a,b) if the parameters m, p, v are given.

To account for memory effects in economic growth models, it is necessary to use a generalization of the equation (1.1), which describes the relationship between investment $I(t)$ and output $Y(t)$. In the work of Tarasova and Tarasov (2016), the authors use the Caputo fractional derivative to develop a generalization for the equation of the accelerator with memory (1.1) in the form

$${}^{C}D^{\alpha}_{0^+}Y(t) = \frac{1}{v} I(t)$$

and then proposed an economic growth model with the involvement of the effects of memory loss as follows

$${}^{C}D^{\alpha}_{0^+}Y(t) - \frac{m(p-a)}{v} Y(t) = -\frac{mb}{v} \tag{2.3}$$

where ${}^{C}D^{\alpha}_{0^+}Y(t)$ is the Caputo fractional derivative of order $\alpha > 0$ of function $Y(t)$.

2.2. Hypothesis development

The following notations will be used: $Y(t)$ is a function that describes the value of output at time t, the number of products produced during time t process production; $I(t)$ is a function that describes the net investment, that is the investment being used for the production expansion.

The assumptions on the economic natural growth with memory:

+ $I(t) = m(pY(t) - C(t))$: the value of the net investment is a fixed part of the profit, which is equal to the difference between the income of $pY(t)$ and the costs $C(t)$, where m is the rate of net investment $0 < m < 1$, i.e. the share of profit, which is spent on the net investment (Tarasova & Tarasov, 2016).

+ $C(t) = aY(t) + b$: the costs $C(t)$ are linearly dependent on the output $Y(t)$, where a is the marginal costs, and b is the independent costs, the part of the cost which does not depend on output value (Tarasova & Tarasov, 2016).

+ We assume that

$$Y(t)=\int_0^t H(t,\tau)I(\tau)d\tau \tag{2.4}$$

of which:

$$H(t,\tau)=\frac{M}{\Gamma(\alpha)}\frac{\psi'(\tau)}{(\psi(t)-\psi(\tau))^{1-\alpha}},\quad t>\tau$$

with $\alpha > 0$, M is a positive real number, $\psi \in C^1[0,\infty)$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in (0,\infty)$.

The kernel $H(t,\tau)$ in (2.4) is called the ψ-power-law fading memory function. In the particular case $\psi(t) = t$, function $H(t,\tau)$ yields the power-law fading memory function that was introduced by Tarasova and Tarasov (2016).

3. Research Methodology

In this section, we provide some basic definitions of fractional calculus which are used further in the present work.

3.1. Fractional calculus

Let $[a,b]$ be a finite interval of the real line ~ and $\alpha > 0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t) \neq 0$ on (a,b). The left-sided fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by (Kilbas et al., 2006).

$$I_{a^+}^{\alpha;\psi}f(t)=\int_a^t\frac{(\psi(t)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\psi'(\tau)f(\tau)d\tau$$

where the function $\Gamma(\cdot)$ denotes the Gamma Euler function and $\alpha > 0$. Noted that the case $\psi(t) = t$ gives $I_{a^+}^{\alpha;\psi}f(t)=I_{a^+}^{\alpha}f(t)$ which is the Riemann-Liouville integral.

The left ψ-Caputo fractional derivative of function f of order $\alpha > 0$ is defined as follows.

Let f and ψ two functions are n-times continuously differentiable functions on $[a,b]$ such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left ψ-Caputo fractional derivative of the function f of order $\alpha > 0$ is defined as the following form (Kilbas et al., 2006).

$${}^{C}D_{a^+}^{\alpha;\psi}f(t)=I_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$$

Note that, if $\alpha = n \in$ •, $\psi(t) = t$, and the usual derivative $f^{(n)}(t)$ of order n exist then ${}^{C}D_{a^+}^{\alpha;\psi}f(t)$ coincides with $f^{(n)}(t)$. If $\psi(t) = t$, then ${}^{C}D_{a^+}^{\alpha;\psi}f(t)$ coincides with the well-known Caputo fractional derivative ${}^{C}D_{a^+}^{\alpha}f(t)$.

Given a function $f \in C^1[a,b]$, that is function f is continuously differentiable on $[a,b]$, and $\alpha > 0$, we have the operator ${}^{C}D_{a^+}^{\alpha;\psi}$ is left inverse to the operator $I_{a^+}^{\alpha;\psi}$:

$${}^{C}D_{a^+}^{\alpha;\psi}\left(I_{a^+}^{\alpha;\psi}f(t)\right)=f(t),\quad \alpha>0 \tag{3.1}$$

The operators $I_{a^+}^{\alpha;\psi}$ and ${}^{C}D_{a^+}^{\alpha;\psi}$ are linear operators.

We recall the two-parametric Mittag-Leffler function (Gorenflo et al., 2020) which plays a crucial role in studying fractional differential equations

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

where $\alpha > 0$, $\beta \in$ ~, and $z \in$ ~. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is a generalization of the classical exponential function e^z. Namely, one can observe that the exponential function is obtained when $\alpha = 1$ and $\beta = 1$, that is $E_{1,1}(z) = e^z$. We also obtain several elementary functions from the definition of Mittag-Leffler function $E_{\alpha,\beta}$ as follows:

$$E_{1,2}(z)=\frac{e^z-1}{z},\ E_{2,1}(z)=\cosh\sqrt{z},\ E_{2,2}(z)=\frac{\sinh\sqrt{z}}{z}$$

The following properties of the Mittag-Leffler functions will be useful in studying fractional-order models:

$$E_{\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}+zE_{\alpha,\alpha+\beta}(z) \tag{3.2}$$

for $\alpha, \beta > 0, z \in$ ~, and

$$\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\lambda \tau^\alpha) d\tau = t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha) \quad (3.3)$$

Combing (3.2) and (3.3) we obtain

$$\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\lambda \tau^\alpha) d\tau = \frac{1}{\lambda}\left(E_{\alpha,1}(\lambda t^\alpha) - 1\right) \quad (3.4)$$

3.2. Fractional-order models

Let $\alpha \in (n-1, n]$, for $n \in$ •, $n \geq 1$, $g(t)$ is a given real function defined on ~$_+$ and $\lambda \in$ ~.

Let $\psi \in C^1[0,T]$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in [0,T]$. We consider the following fractional differential equation

$$^C D_{0^+}^{\alpha;\psi} y(t) - \lambda y(t) = g(t), \quad 0 < t \leq T, \quad (3.5)$$

with initial condition

$$y(0) = c_0, y_\psi^{(j)}(0) = c_j, \quad j = 2,..., n-1 \quad (3.6)$$

where $y_\psi^{(j)}(t) = \left(\frac{1}{\quad}\frac{d}{\quad}\right)^j y(t)$ and $\left(\frac{d}{dt}\right)^j y(t)$ is the usual derivative of order j of function y. For the given Mittag-Leffler function $E_{\alpha,\beta}(x)$ and $\lambda \in$ ~, we define the function $E_{\alpha,\beta}(\lambda;x)$ by

$$E_{\alpha,\beta}(\lambda;x) := x^{\beta-1} E_{\alpha,\beta}(\lambda x^\beta)$$

and we denote

$$\mathrm{E}_{\alpha,\beta}(\lambda;x) *_\psi g(t) := \int_0^t \mathrm{E}_{\alpha,\beta}(\lambda;\psi(\tau)) g\left(\psi^{-1}(\psi(t) - \psi(\tau))\right) \psi'(\tau) d\tau$$

The solution to the problem (3.5) - (3.6) is given by

$$y(t) = \sum_{j=0}^{n-1} c_j (\psi(t))^j E_{\alpha,j+1}(\lambda(\psi(t))^\alpha) + \mathrm{E}_{\alpha,\beta}(\lambda;\psi(t)) *_\psi g(t)$$

We now consider a particular case of the problem (3.5) - (3.6), that is $g(t)$ and $C \in$ ~. In this case,

$$\mathrm{E}_{\alpha,\beta}(\lambda;x) *_\psi g(t) = C \int_0^t \mathrm{E}_{\alpha,\beta}(\lambda;\psi(\tau)) \psi'(\tau) d\tau$$

Together with (3.4), we have

$$\mathrm{E}_{\alpha,\beta}(\lambda;x) *_\psi g(t) = \frac{C}{\lambda}\left(E_{\alpha,1}(\lambda(\psi(t))^\alpha) - 1\right)$$

Thus, the problem (3.5) - (3.6) in the special case $g(t) = C$ has a unique solution

$$y(t) = \sum_{j=0}^{n-1} c_j (\psi(t))^j E_{\alpha,j+1}(\lambda(\psi(t))^\alpha) + \frac{C}{\lambda}\left(E_{\alpha,1}(\lambda(\psi(t))^\alpha) - 1\right) \quad (3.7)$$

4. Result and discussion

4.1. Result

This subsection delved into modeling the Harrod-Domar model using a fractional calculus approach, specifically the ψ-Caputo fractional derivative. Since (2.4), we get $Y(t) = M(I_{0^+}^{\alpha;\psi} I)(t)$. By (3.1),

$$^C D_{0^+}^{\alpha;\psi} Y(t) = \frac{1}{v} I(t), \quad v = \frac{1}{M} \quad (4.1)$$

We now apply this equation to studying the relationship between the net investment $I(t)$ and the marginal value of the output of order $\alpha \in (n-1, n]$ for $n \in$ •. Namely, combining (4.1) with the assumptions $I(t) = m(pY(t) - C(t))$ and $C(t) = aY(t) + b$, we deduce that

$$^C D_{0^+}^{\alpha;\psi} Y(t) - \frac{m(p-a)}{v} Y(t) = -\frac{mb}{v} \quad (4.2)$$

Equation (4.2) is a Harrod-Domar fractional differential equation with ψ-Caputo fractional derivative of order $\alpha > 0$. The new point in the equation (4.2) is the replacement of the classical derivative by the ψ-Caputo derivative. If $\alpha = 1$ and $\psi(t) = t$ then (4.2) reduces the natural growth model without memory effect. According to (3.7), the equation (4.2) has solution $Y(t)$ is expressed in terms of the Mittag-Leffler functions as follows

$$Y(t) = \sum_{j=0}^{n-1} Y^{(j)}(0)(\psi(t))^j E_{\alpha,j+1}(\lambda(\psi(t))^\alpha) + \frac{C}{\lambda}\left(E_{\alpha,1}(\lambda(\psi(t))^\alpha) - 1\right) \quad (4.3)$$

where $Y_{\psi}^{(j)}(0)$ are the values of the derivatives $Y_{\psi}^{(j)}(t)$ of $Y(t)$ of the order j at $t = 0$, $\lambda = \frac{m(p-a)}{v}$ and $C = -\frac{mb}{v}$. This solution reports the output $Y(t)$ in the framework of the natural growth model with ψ-power-law fading memory. We consider the solution of the equation (4.2) in some special cases $0 < \alpha \le 1$ and $1 < \alpha \le 2$.

In the case $1 < \alpha \le 2$: We have

$$Y(t) = Y(0)E_{\alpha,1}\left(\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right) + \frac{b}{p-a}\left(1 - E_{\alpha,1}\left[\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right]\right)$$

In the case $1 < \alpha \le 2$: We have

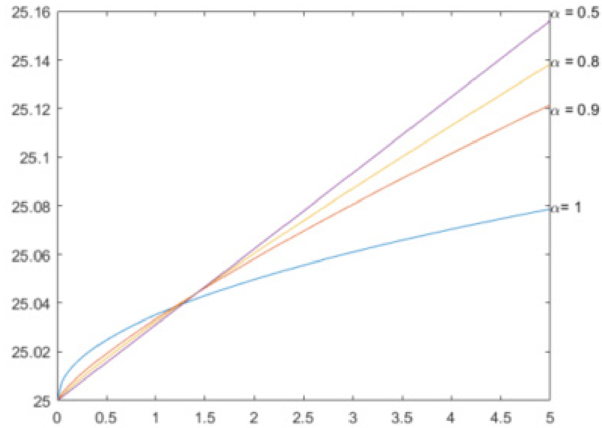
$$Y(t) = Y(0)E_{\alpha,1}\left(\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right) + Y'(0)\psi(t)E_{\alpha,2}\left(\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right) + \frac{b}{p-a}\left(1 - E_{\alpha,1}\left[\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right]\right)$$

In the following numerical examples, we show the behavior of the output $Y(t)$, when we take into account the effects of the ψ-power-law memory, in the form of graphs. To evaluate the solutions $Y(t)$, we computed the Mittag-Leffler function $E_{\alpha,\beta}(x)$ by Matlab code ml that was introduced by Garrappa (2014, 2015).

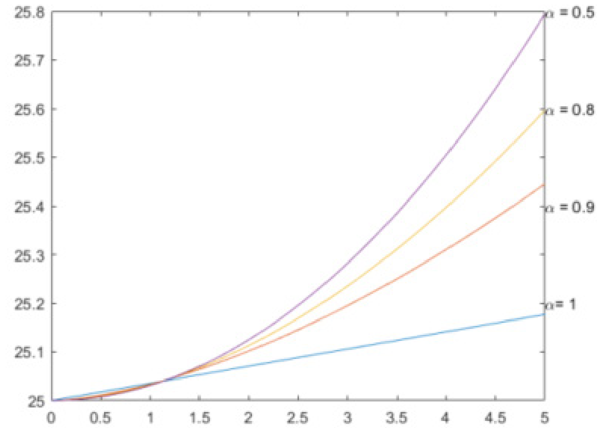
We first consider the solution to the equation (4.2) with the initial value condition $Y(0) = 25$, model parameters are $a = 0.2$, $b = 3$, $p = 0.6$, $m = 0.2$, $v = 45$.

$$Y(t) = \frac{b}{p-a}\left(1 - E_{\alpha,1}\left[\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right]\right) + Y(0)E_{\alpha,1}\left(\frac{m(p-a)}{v}(\psi(t))^{\alpha}\right) = 7.5 + 17.5E_{\alpha,1}\left[\frac{2}{1125}(\psi(t))^{\alpha}\right] \quad (4.4)$$

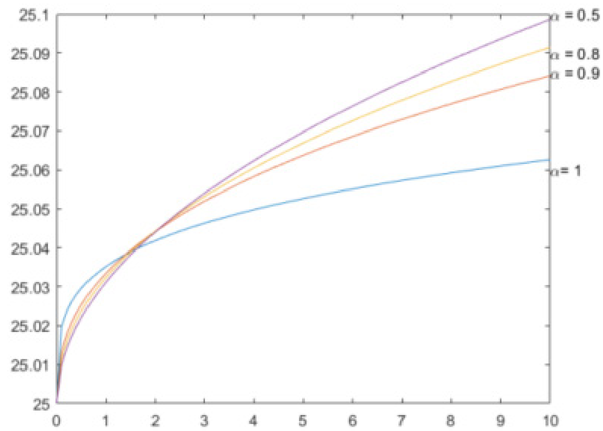
Figure 1 presents the solutions to the problem (4.2) for different values of α: $\alpha = 0.5$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1$ and for three functions $\psi(t) = t$, $\psi(t) = t^2$, $\psi(t) = \sqrt{t}$.



(a)



(b)



(c)

Figure 1. Plots of $Y(t)$ for different values of $\alpha \in (0,1]$. (a) Plot of $Y(t)$ for $\psi(t) = t$. (b) Plot of $Y(t)$ for $\psi(t) = t^2$. (c) Plot of $Y(t)$ for $\psi(t) = \sqrt{t}$.

Secondly, we consider the solutions to the equation (4.2) with the initial value conditions $Y(0) = 25$, $Y'_\psi(0) = 0.2$ model parameters are $a = 0.2$, $b = 3$, $p = 0.6$, $m = 0.2$, $v = 45$. We have

$$Y(t) = 7.5 + 17.5 E_{\alpha,1}\left[\frac{48}{225}(\psi(t))^{\alpha}\right] + 0.2\psi(t) E_{\alpha,2}\left[\frac{2}{1125}(\psi(t))^{\alpha}\right]$$

Figure 2 presents the solutions to problem (4.2) for different values of α: $\alpha = 1.8$, $\alpha = 1.5$, $\alpha = 1.3$, $\alpha = 1$, and for three functions $\psi(t) = t$, $\psi(t) = t^2$, $\psi(t) = \sqrt{t}$.

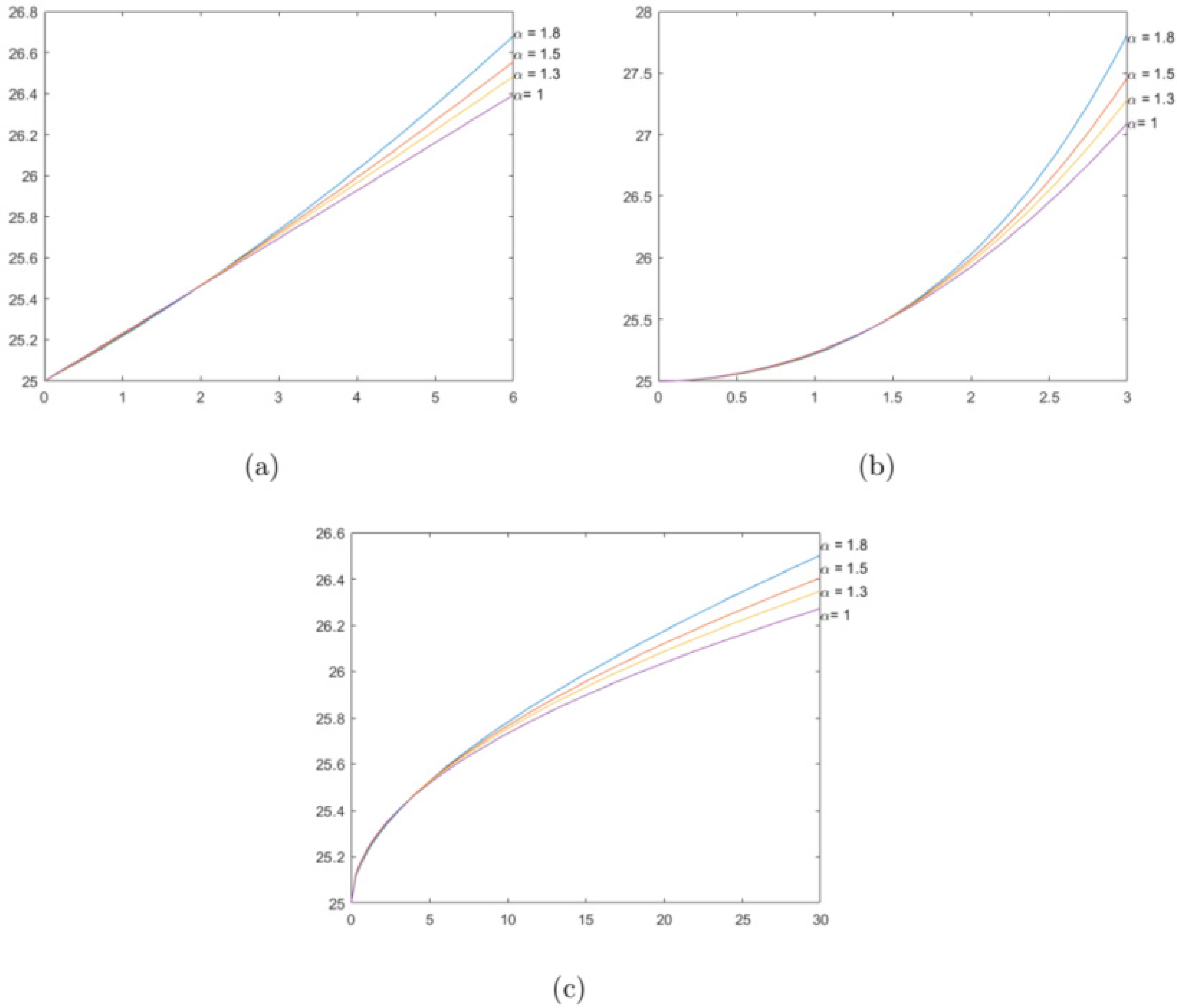


Figure 2. Plots of $Y(t)$ for different values of $\alpha \in (1,2]$. (a) Plot of $Y(t)$ for $\psi(t) = t$. (b) Plot of $Y(t)$ for $\psi(t) = t^2$. (c) Plot of $Y(t)$ for $\psi(t) = \sqrt{t}$.

4.2. Discussion

In our proposed model, the assumptions on the value of the net investment and the costs $C(t)$ are the same as in Tarasova and Tarasov (2016). However, our assumption on the kernel $H(t,\tau)$ in (2.4) is a generalization of the power-law fading memory function that was introduced by Tarasova and Tarasov (2016). Moreover, the ψ-Caputo derivative of the positive $\alpha > 0$ coincides with the Caputo derivative. For this reason, our proposed fractional-order models are also a generalization of the economic growth models with fractional derivatives in Tarasova and Tarasov (2016).

As a result, in the case $0 < \alpha < 1$, the solution $Y(t)$ in $[T, \infty)$ (for large enough T) of the model with ψ-power-law fading memory is slower growth in comparison with the standard model without memory (in the case $\alpha = 1$); while $1 < \alpha < 2$ the solution $Y(t)$ in $[T, \infty)$ (for large enough T) of the model with ψ-power-law fading memory is faster growth, compared with the standard model without memory.

We can see that the behavior of the output function $Y(t)$ is essentially dependent on the attendance or absence of memory effects and function $\psi(t)$. Therefore, accounting for memory effects can introduce new types of behavior for the same parameters of economic growth models. In studying economic growth models, we need to take into account the memory effect in these models. In general, ignoring memory effects in macroeconomic models can lead to qualitatively different conclusions.

5. Conclusions

In this study, we propose a class of Harrod-Domar fractional differential equations which provides a more advanced framework for understanding economic growth, incorporating the effects of time and historical context into the relationship between investment and output. While more complex, it can yield insights that align more closely with the dynamic nature of real economies. Furthermore, unlike the usual derivative, fractional derivative operators are non-local and possess a memory effect. Hence, this approach allows for greater flexibility in modeling economic behavior, making it possible to account for non-instantaneous effects and the cumulative impact of previous decisions. In the future, we will try to study some other economic growth models, such as Keynes model Ramsey mode, Haavelmo mode, etc.

Implications

In this work, we considered an economic growth model in which the class of fading of memory was characterized by the fractional order $\alpha > 0$ and the function ψ. In real economies, memory may be different for different types of economic agents and different cases. In these situations, we can consider applying the kernel of the fractional integral operator in the form of a ψ-power-law fading memory function.

Limitations and future research directions

In view of practical applications, the assumptions on economic natural growth with memory have some limitations, such as the cost function $C(t)$ in the form of quadratic, cube, and even nonlinear forms. Hence, the nonlinear fractional differential equation can be regarded as a more feasible model equation than that we have studied in the current work, but it will be more difficult. We expect to construct results for this more generalized case in the future.

References

Garrappa, R. (2014). The Mittag-Leffler function. *MATLAB Central File Exchange*, *201*. http://www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function

Garrappa, R. (2015). Numerical evaluation of two and three parameter Mittag-Leffler functions. *SIAM Journal on Numerical Analysis*, *53*(3), 1350-1369. https://doi.org/10.1137/140971191

Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. V. (2020). *Mittag-Leffler functions, Related topics and applications* (p. 540). Berlin: Springer. https://doi.org/10.1007/978-3-662-43930-2

Granger, C. W. J., & Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis*, *1*(1), 15-29. https://doi.org/10.1111/j.1467-9892.1980.tb00297.x

Kee, C. Y., Chua, C., Zubair, M., & Ang, L. K. (2022). Fractional modeling of urban growth with memory effects. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(8). https://doi.org/10.1063/5.0085933

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations: Volume 204 of North-Holland Mathematics Studies*. Elsevier.

Tarasov, V. E. (2020). Fractional econophysics: Market price dynamics with memory effects. *Physica A: Statistical Mechanics and its Applications*, *557*. https://doi.org/10.1016/j.physa.2020.124865

Tarasov, V. E. (2021). Nonlinear growth model with long memory: generalization of Haavelmo model. *Nonlinear Dynamics*, *104*(4), 4413-4425. https://doi.org/10.1007/s11071-021-06484-2

Tarasova, V. V., & Tarasov, V. E. (2016). Fractional dynamics of natural growth and memory effect in economics. *European Research*, *12*(23), 30-37. https://doi.org/10.20861/2410-2873-2016-23-004

Tarasova, V. V., & Tarasov, V. E. (2017). Logistic map with memory from economic model. *Chaos, Solitons & Fractals*, *95*, 84-91. https://doi.org/10.1016/j.chaos.2016.12.012

Tarasova, V. V., & Tarasov, V. E. (2018). Concept of dynamic memory in economics. *Communications in Nonlinear Science and Numerical Simulation*, *55*, 127-145. https://doi.org/10.1016/j.cnsns.2017.06.032

Tarasova, V. V., & Tarasov, V. E. (2018). Dynamic intersectoral models with power-law memory. *Communications in Nonlinear Science and Numerical Simulation*, *54*, 100-117. https://doi.org/10.1016/j.cnsns.2017.05.015

Tejado, I., Pérez, E., & Valério, D. (2019). Fractional calculus in economic growth modelling of the group of seven. *Fractional Calculus and Applied Analysis*, *22*, 139-157. https://doi.org/10.1515/fca-2019-0009

Traore, A., & Sene, N. (2020). Model of economic growth in the context of fractional derivative. *Alexandria Engineering Journal*, *59*(6), 4843-4850. https://doi.org/10.1016/j.aej.2020.08.047